# ELECTROMAGNETO-QUASISTATIC VERSUS FULL MAXWELL MODELS FOR LOW- AND MIDDLE-FREQUENCY FIELD PROBLEMS

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## Abstract

This paper provides a discussion of the applications of electromagneto-quasistatic (EMQS) field formulations of Darwin-type, where radiation effects are neglected, and the regular full Maxwell field formulations with respect to their usability in low- and middle-frequency field problems. It is shown that both types of formulation result in ill-conditioned systems of equations, where aspects of numerical stability and solver efficiency are affected.

## 1 Introduction

So-called middle-frequency field problems must be considered, for example, when simulating plasma or EMC analysis problems. These scenarios belong to the quasistatic regime, where radiation effects can be neglected because the size of the objects under consideration is small compared to the wave lengths involved. However, if capacitive, inductive and possibly resistive effects need to be considered simultaneously, traditional quasistatic formulations may fail and electromagneto-quasistatic (EMQS) field models related to the original Darwin model [1] have become of interest. They are not gauge invariant; i.e., depending on the choice of the gauge equations, the resulting electromagnetic field solutions may differ as approximations to those of the full Maxwell equations. This difference can be large, as demonstrated in [2], where it is shown that in certain circumstances, Darwin models do not account for the interaction between inductive and capacitive effects, depending on the gauge. On the other hand, Maxwell equations are regular and are capable of modeling radiation effects. Either formulation suffers from a numerical low-frequency instability resulting from the null spaces of the curl operators. This requires us to additionally consider low-frequency stabilizations. For practical field simulations of middle-frequency problems, implementations of EMQS compete with those of stabilized full Maxwell formulations.

## 2 Maxwell Equations and EMQS Field Models

The Maxwell equations for electromagnetic field phenomena formulated in terms of a magnetic vector potential $\boldsymbol{A}$ and the scalar electric potential $\varphi$ for the electric field $\boldsymbol{E} = -\partial_t \boldsymbol{A} - \nabla\varphi$ and the magnetic induction $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ can be represented by the original Maxwell-Ampère equation and the continuity equation

$$\nabla \times (\nu \nabla \times \boldsymbol{A}) + \kappa\partial_t \boldsymbol{A} + \kappa\nabla\varphi + \varepsilon\nabla\partial_t\varphi + \varepsilon\partial_{tt}\boldsymbol{A} = \boldsymbol{j}_{\mathrm{s}} \quad (1)$$

$$\nabla \cdot (\kappa\partial_t \boldsymbol{A} + \kappa\nabla\varphi + \varepsilon\nabla\partial_t\varphi + \varepsilon\partial_{tt}\boldsymbol{A}) = 0 \quad (2)$$

with a given source current density $\boldsymbol{j}_{\mathrm{s}}$. All EMQS approximations relate to the original Darwin model [1] by omission of the wave propagation term $\varepsilon\partial_{tt}\boldsymbol{A}$ in (1)

$$\nabla \times (\nu \nabla \times \boldsymbol{A}) + \kappa\partial_t \boldsymbol{A} + \kappa\nabla\varphi + \varepsilon\nabla\partial_t\varphi = \boldsymbol{j}_{\mathrm{s}}. \quad (3)$$

Additional gauge equations specify the different approximations: While the original Darwin model uses a Coulomb gauge $\nabla \cdot \boldsymbol{A} = 0$ and the Poisson equation for the electric scalar potential, later EQMS formulations involve the Darwin-continuity equation

$$\nabla \cdot (\kappa\partial_t \boldsymbol{A} + \kappa\nabla\varphi + \varepsilon\nabla\partial_t\varphi) = 0. \quad (4)$$

in conjunction with an additional Coulomb-type gauges that are either added to (4) [3] or considered in grad-div expression augmentations to (3) [4]. An implicit gauge is achieved by combining Darwin-Ampère's equation (3) with the full Maxwell continuity equation (2), which also results in symmetric discrete formulations and maintains a port-Hamiltonian structure of the full set of Maxwell equations [5]. To avoid numerical instabilities, EMQS formulations as in [6] may additionally enforce the Coulomb-type gauge $\nabla \cdot (\varepsilon\partial_t\boldsymbol{A}) = 0$ with Lagrange-multipliers similar to [7]. The split-scalar potential approach to low-frequency stabilization introduced for the full set of Maxwell equations [8] is also used within an EMQS formulation in [9]. These formulations represent variants of the Lagrange multiplier penalty approach, as shown in [5]. Alternative EMQS formulations as e.g. in [10] rely on a domain decomposition approach introducing different Coulomb-type gauges $\nabla \cdot (\kappa\boldsymbol{A}) = 0$ in conducting and $\nabla \cdot (\varepsilon\partial_t\boldsymbol{A}) = 0$ non-conducting areas. This can be related to domain-wise tree-cotree decompositions, e.g. [11, 12].

## 3 Frequency Domain

Let us investigate the Lagrange multiplier low-frequency stabilized full Maxwell formulation with wave propagation $\mathbf{W} = -\omega^2\mathbf{M}_\varepsilon$ and the corresponding EMQS formulation in

| | $\mathbf{C}^\top\mathbf{M}_{\nu_0}\mathbf{C}$ | $\omega\mathbf{M}_\kappa$ | $\mathbf{M}_\kappa\mathbf{G}$ | $\omega\mathbf{M}_\varepsilon\mathbf{G}$ | $\omega^2\mathbf{M}_\varepsilon(=-\mathbf{W})$ |
|---|---|---|---|---|---|
| $f$ | $\nu_0 h^{-1}$ | $\omega\kappa h$ | $\kappa h$ | $\omega\varepsilon h$ | $\omega^2\varepsilon h$ |
| $10^0$ Hz | $O(10^9)$ | $O(10^5)$ | $O(10^4)$ | $O(10^{-13})$ | $O(10^{-12})$ |
| $10^3$ Hz | $O(10^9)$ | $O(10^8)$ | $O(10^4)$ | $O(10^{-10})$ | $O(10^{-6})$ |
| $10^6$ Hz | $O(10^9)$ | $O(10^{11})$ | $O(10^4)$ | $O(10^{-7})$ | $O(10^0)$ |
| $10^9$ Hz | $O(10^9)$ | $O(10^{14})$ | $O(10^4)$ | $O(10^{-4})$ | $O(10^6)$ |

Table 1: Magnitudes of matrix entries.

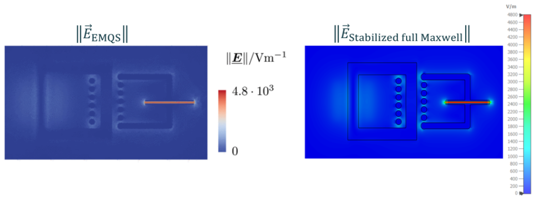


Figure 1: Electric field of transformer [4]. EMQS solution (left). Stabilized full Maxwell solution (CST Suite) (right).

[6] with $\mathbf{W}=0$. Using a matrix-vector-notation related to the finite integration technique (FIT), this reads

$$\begin{bmatrix} \mathbf{W}+j\omega\mathbf{M}_\kappa+\mathbf{C}^\top\mathbf{M}_\nu\mathbf{C} & \mathbf{M}_\sigma\mathbf{G} & j\omega\mathbf{M}_\varepsilon\mathbf{G} \\ \mathbf{G}^\top\mathbf{M}_\sigma & \frac{1}{(j\omega)}\mathbf{G}^\top\mathbf{M}_\sigma\mathbf{G} & 0 \\ j\omega\mathbf{G}^\top\mathbf{M}_\varepsilon & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \varphi \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{j}_\mathrm{s} \\ \frac{1}{(j\omega)}\mathbf{G}^\top\mathbf{j}_\mathrm{s} \\ 0 \end{bmatrix},$$

where $\omega=2\pi f$ is the angular frequency, $\mathbf{C},\mathbf{G}$ are discrete curl and gradient matrices with entries in $\{+1,-1,0\}$, $\mathbf{M}_\sigma:=[\mathbf{M}_\kappa+j\omega\mathbf{M}_\varepsilon]$ with $\mathbf{M}_\kappa,\mathbf{M}_\varepsilon,\mathbf{M}_\nu$ discrete material matrices for electric conductivity $\kappa$, permittivity $\varepsilon$ and reluctivity $\nu$, respectively, $\mathbf{a},\varphi$, the degree of freedom vectors, $\lambda$ the Lagrange multiplier vector, $\mathbf{j}_\mathrm{s}$ the vector of source currents.

Table 1 lists the order of magnitude of the entries of the various matrix blocks for a simulation problem with an assumed uniform grid spacing $h=10^{-3}$ m, material parameters $\kappa:=10^7$ S/m, $\nu_0=1/\mu_0=1/(4\pi\cdot10^{-7})$ Vs/(Am), $\varepsilon=8,854\cdot10^{-12}$ As/(Vm).

In the full paper, a more detailed comparison of time-domain EMQS and full Maxwell formulations will be given based on the following questions:

i) Correctness: Does an EMQS formulation contain all relevant effects?
ii) Validity: When does wave propagation become non-negligible?
iii) Efficiency: What formulation is more convenient to solve directly/iteratively w.r.t. the number of degrees of freedom, symmetry, definiteness, condition number, sparsity, etc.?

## 4 Numerical Example

A transformer model [4] is used to compare a grad-div stabilized EMQS formulation [4] with a stabilized full Maxwell formulation [12]. Both solutions ($f=1$ MHz) are depicted in Fig. 1. They agree within the given accuracy, i.e., the EMQS formulation contains all relevant effects, see i), in this case.

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